Comment on "Connection between the Burgers equation with an elastic forcing term and a stochastic process"

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In the above mentioned paper by E. Moreau and O. Vallée [Phys. Rev. E **73**, 016112 (2006)], the one-dimensional Burgers equation with an elastic (attractive) forcing term has been claimed to be connected with the Ornstein-Uhlenbeck process. We point out that this connection is valid only in the case of repulsive forcing.

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Let us consider the Langevin equation for the one-dimensional stochastic process in the external conservative force field $F(x)=-dV(x)/dx$,

$$\frac{dx}{dt} = F(x) + \sqrt{2\nu}b(t), \tag{1}$$

where $b(t)$ stands for the normalized white noise $\langle b(t)\rangle=0$, $\langle b(t')b(t)\rangle=\delta(t-t')$. The corresponding Fokker-Planck equation for the probability density $\rho(x,t)$ reads

$$\partial_t\rho = \nu\partial_{xx}\rho - \partial_x(F\rho), \tag{2}$$

and by means of a standard substitution

$$\rho(x,t) = \Psi(x,t)\exp[-V(x)/2\nu],$$

Ref. [1], can be transformed into the generalized diffusion equation for an auxiliary function $\Psi(x,t)$,

$$\partial_t\Psi = \nu\partial_{xx}\Psi - \mathcal{V}(x)\Psi, \tag{3}$$

where

$$\mathcal{V}(x) = \frac{1}{2}\left(\frac{F^2}{2\nu} + \partial_x F\right). \tag{4}$$

As discussed in detail in Refs. [2–4], given the so-called forward drift $b(x,t)$ of the Markovian diffusion process, in the above identified with $b(x,t)\doteq F(x)$ one readily infers the so-called backward drift of this process,

$$b_\star(x,t) \doteq b(x,t) - 2\nu\partial_x(\ln\rho)(x), \tag{5}$$

which is known to solve the forced Burgers equation

$$\partial_t b_\star + b_\star\partial_x b_\star - \nu\partial_{xx}b_\star = \mathcal{F}, \tag{6}$$

where $\mathcal{F}=+2\nu\partial_x\mathcal{V}$.

For the Ornstein-Uhlenbeck process $b(x)=F(x)=-\kappa x$ and accordingly

$$\mathcal{V}(x) = \frac{\kappa^2x^2}{4\nu} - \frac{\kappa}{2}. \tag{7}$$

Substituting the inferred $\mathcal{V}(x)$ into Eq. (3), we obtain Eq. (32) of Ref. [5]. We observe that the velocity field $u(x,t)$, defined by Eq. (35) of Ref. [5], does coincide with our $b_\star(x,t)$, provided we set $b(x,t)=-\kappa x$ in Eq. (5).

The related $\mathcal{V}(x)$ gives rise to

$$\mathcal{F}(x) = 2\nu\partial_x\mathcal{V}(x) = +\kappa^2 x, \tag{8}$$

while an original elastic forcing problem addressed by Ref. [5], Eq. (2) therein, has the form

$$\partial_t u + u\partial_x u - \nu\partial_{xx}u = -\kappa^2 x \tag{9}$$

and clearly differs from Eq. (6), with the necessarily arising $+\kappa^2 x$ on its right-hand side, by an innocent looking but crucial in the present context sign of the forcing term.

Let us add that this particular sign issue has received due attention in Ref. [6]. A specific class of diffusion-type processes has been considered that would account for standard Newtonian accelerations [of the form $-\partial_x W$ with $W(x)$ bounded from below] on the right-hand side of Eq. (6). It is in principle possible at the price of introducing an additional pressure-type forcing term.

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[1] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1989).
[2] P. Garbaczewski and G. Kondrat, Phys. Rev. Lett. **77**, 2608 (1996).
[3] P. Garbaczewski, G. Kondrat, and R. Olkiewicz, Phys. Rev. E **55**, 1401 (1997).
[4] P. Muratore Ginanneschi, Physica D **115**, 341 (1998).
[5] E. Moreau and O. Vallée, Phys. Rev. E **73**, 016112 (2006).
[6] P. Garbaczewski, Phys. Rev. E **57**, 569 (1998).

